Three-sided matchings and separable preferences

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Abstract

In this paper we provide sufficient conditions for the existence stable matchings for three-sided systems.

Introduction: The two-sided matching model of Gale and Shapley (1962) can be interpreted as one where a non-empty finite set of firms need to employ a non-empty finite set of workers. Further, each firm can employ at most one worker and each worker can be employed by at most one firm. Each worker has preferences over the set of firms and each firm has preferences over the set of workers. An assignment of workers to firms is said to be stable if there does not exist a firm and a worker who prefer each other to the ones they are associated with in the assignment. Gale and Shapley (1962) proved that every two-sided matching problem admits at least one stable matching.

In this paper we extend the above model by including a non-empty finite set of techniques. A technique can be likened to a machine that is owned by a technologist who is neither a firm nor a worker, and which the firm and worker together use for production. Further each technologist owns exactly one technique. Each firm has preferences over the set of ordered pairs of workers and techniques, each worker has preferences over the set of ordered pairs of firms and techniques and each technologist has preferences over ordered pairs of firms and workers. Such models [see Alkan (1988)] are called three-sided systems. A matching in a three-sided system consists of disjoint triplets, each triplet comprising a firm, a worker and a technologist. A stable matching for a three-sided system is a matching which does not admit a triplet whose members are better off together than at their current designations. Alkan (1988) provided an example of a three-sided system that does not admit a stable matching. Danilov (2003) established the existence of a stable matching for lexicographic three-sided systems.

The preference of a firm is separable if its preference over workers is independent of the technique and its preference over techniques is independent of the worker. The preference of a worker is separable if its preference over firms is independent of the technique and its preference over techniques is independent of the firm. A three-sided system is said to be separable if preferences of all firms and workers are separable. Through out the paper, we assume that the preferences of the workers are separable between firms and techniques. A special case of such preferences is lexicographic preferences, with firms enjoying priority over techniques. If in addition the preferences of the firms are lexicographic (without necessarily being separable), with workers enjoying priority over techniques, then the system is called lexicographic. Lexicographic systems are clearly separable.

In this paper we show that if a three-sided system is lexicographic for workers and satisfies a property called *Technical Specialization* then there exists a stable matching. Technical Specialization says: given two distinct firm-worker pairs, the technique that is best for the firm in one pair is different from the technique that is best for the firm in the other. Note that the discrimination property is strictly stronger than the weak discrimination property that we discussed earlier. We also provide an example of a three-sided system with preferences of workers being both lexicographic as well as separable, that does not admit a stable matching. In this

example the preferences of the firms are neither lexicographic nor separable.
Neither technical specialization nor the proof of theorem that establishes the existence of a stable matching when technical specialization is satisfied by a three-sided system, takes cognizance of the preferences of the technologists. In a way, the stable matching that is obtained may have resulted by ‘coercing’ the technologists. While this may make the technical specialization an unpalatable assumption, it is worth remembering, that a stable matching for a three-sided system, does not require that every side of the system play an active role in determining its viability.
Alternatively one may assume that the three-sided system is strongly separable i.e. lexicographic for workers and separable for firms. In such a scenario we need to assume that the preferences of firms and technologists over workers are in “agreement” (i.e. given a firm, a technologists ranks the workers in the same way that the firm does) to show that a three-sided system admits a stable matching. Agreement over workers in a strongly separable environment implies some kind of a hierarchy where the worker cares only about the firm and forms the bottom layer, whereas the technologist’s preferences over the workers “echoes” the preferences of the firm it is engaged with.
Following the tradition of Gale and Shapley (1962), we model our analysis in terms of a firm employing at most one worker. By present day reckoning, a firm employing at most one worker is usually a small road-side shop, rather than an industrial unit. Hence, it might appear as if our analysis has little if no relevance to the more common real world situations. However, it may well be a reasonable starting point for the cooperative theory of multi-sided systems. Roth and Sotomayor (1988) contain an elaborate discussion of matching models, where firms may employ more than one worker. It turns out in their analysis, that the cooperative theory for such firms is almost identical to the cooperative theory arising out of the Gale and Shapley (1962) framework. This occurs, since each firm can be replicated as often as the number of

workers it can employ, with each replica having the same preferences over workers as the original firm. Further, the preferences of the workers between replicas of two different firms should be exactly the same as her preferences between the originals. On the other hand, the non-cooperative theory where each firm employs more than one worker is considerably different from the non-cooperative theory where firms may employ at most one. It is noteworthy that the cooperative theory for many-to-many two-sided matching models does not permit the same replication argument. This has been shown in Lahiri (2006).

The analysis reported in this paper, attempts at extending results pertaining to the existence of stable matchings in a labor market, by introducing technology as an essential determinant of the results that we obtain. Since our paper, is concerned with the cooperative theory of three-sided systems, the model that we use of a firm employing at most one worker, continues to provide valuable insights concerning the existence of stable matchings in labor markets.

The Model: Let W be a non-empty finite set denoting the set of workers, F a non-empty finite set denoting the set of firms and T a non-empty finite set denoting the set of techniques. We assume for the sake of simplicity that the $|T|$ cardinality of T is equal to the number of firms ($|F|$) which in turn is equal to the number of workers ($|W|$).

Each $w \in W$ has preference over $F \times T$ defined by a linear order (i.e. anti-symmetric, reflexive, complete and transitive binary relation) $\geq_w$ whose asymmetric part is denoted $>_w$. Each $f \in F$ has preference over $W \times T$ defined by a linear order $\geq_f$ whose asymmetric part is denoted $>_f$. Each $t \in T$ has preference over $F \times W$ defined by a linear order $\geq_t$ whose asymmetric part is denoted $>_t$.

A three-sided system is given by the array $[\{\geq_f : f \in F\}, \{\geq_w : w \in W\}, \{\geq_t : t \in T\}]$.

A job-matching is a one-to-one function m from F to W. A technique matching is a one-to-one function n from F to T. Since F,T and W all have the same cardinality, every job-matching and every technique matching is of necessity a bijection. A pair (m,n) where m is a job-matching and n is a technique matching is called a matching (for the three-sided system).

A matching (m,n) is said to be **stable** if there does not exist $f \in F$, $w \in W$ and $t \in T$ such that: $(w,t) >_f (m(f),n(f))$, $(f,t) >_w (m^{-1}(w), n(m^{-1}(w)))$ and $(f,w) >_t (m(n^{-1}(t), n^{-1}(t))$.

A three-sided system is said to be **separable for workers** if for all $w \in W$ there exists linear orders P_w on F and Q_w on T such that for all $(f,t), (f',t') \in F \times T$: $(f,t) \geq_w (f',t)$ if and only if fP_wf' and $(f,t) \geq_w (f,t')$ if and only if tQ_wt'
A three-sided system is said to be **separable for firms** if for all $f \in F$ there exists linear orders P_f on W and Q_f on T such that for all $(w,t), (w',t') \in W \times T$: $(w,t) \geq_f (w',t)$ if and only if wP_fw' and $(w,t) \geq_f (w,t')$ if and only if tQ_ft'.
A three-sided system is said to be **separable** if it is separable for <u>both</u> firms and workers.
A separable three-sided system is said to be **lexicographic for workers** if for all $w \in W$ there exists linear orders P_w on F and Q_w on T such that: (a) for all $f,f' \in F$ with $f \neq f'$ and $t,t' \in T$: fP_wf' implies $(f,t) >_w (f',t')$; (b) for all $f \in F$ and $t,t' \in T$ with $t \neq t'$: tQ_wt' implies $(f,t) >_w (f,t')$.
A three-sided system is said to be **lexicographic for firms** if for all $f \in F$ there exists linear orders P_f on W such that for all $w,w' \in W$ with $w \neq w'$ and $t,t' \in T$: wP_fw' implies $(w,t) >_f (w',t')$.

Note that unlike the definition of lexicographic for workers our definition of a three-sided system being lexicographic for firms does not require the system to be separable.

A three-sided system is said to be **lexicographic** if it is both lexicographic for workers as well as for firms.

A three-sided system is said to be **strongly separable** if it is separable for firms and lexicographic for workers.
Hence strongly separable system is separable, since by dint of it being lexicographic for workers it is separable for workers as well.

Danilov (2003) proved that if a three-sided system is lexicographic, then it admits a stable matching.

<u>**Existence of Stable Matchings**</u>: A three-sided system is said to satisfy **Technical Specialization** (TS) if there exists a function β:$F\times W\rightarrow T$ such that (a) for all $w,w_1\in W$ and $f,f_1 \in F$ with $w\neq w_1$ and $f\neq f_1$: $\beta(f,w) \neq \beta(f_1,w_1)$; (b) for all $w\in W$, $f\in F$ and $t\in T$: $(w,\beta(f,w)) \geq_f (w,t)$.

Theorem 1: Suppose a three-sided system that is lexicographic for workers satisfies TS. Then there exists a stable matching.

Proof: Suppose preferences are lexicographic for workers and the system satisfies TC.
Hence for all $w\in W$ there exists linear orders P_w on F and Q_w on T such that: (a) for all $f,f'\in F$ with $f \neq f'$ and $t,t'\in T$: fP_wf' implies $(f,t) >_w (f',t')$; (b) for all $f\in F$ and $t,t'\in T$ with $t \neq t'$: tQ_wt' implies $(f,t) >_w (f,t')$.
For $f\in F$ let P_f be the linear order on W such that for all $w, w'\in W$: wP_fw' if and only if $(w, \beta(f,w)) \geq_f (w', \beta(f, w'))$.
Consider the two-sided matching problem where the preference of a firm f is given by P_f, and the preference of a worker w is given by P_w.
As in Gale and Shapley (1962) we get a stable job-matching m, i.e. for all $w\in W$ and $f\in F$: either $m(f)P_fw$ or $m^{-1}(w)P_wf$.
The technique-matching n is defined as follows:

For all $f \in F$: $n(f) = \beta(f,m(f))$.
By TS, n is well defined.
Suppose the matching (m,n) is not stable. Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that: $(f,t) >_w (m^{-1}(w),n(m^{-1}(w))$, $(w,t) >_f (m(f), n(f))$ and $(f,w) >_t ((n^{-1}(t),m(n^{-1}(t))$.
Let $m^{-1}(w) = f_0$, and $n^{-1}(t) = f_1$.
Since the preferences of workers are lexicographic (with firms receiving priority over techniques), $(f,t) >_w (m^{-1}(w),n(m^{-1}(w))$ implies fP_wf_0.
However since m is stable, fP_wf_0 implies $m(f)P_fw$.
Thus $(m(f), n(f)) = (m(f), \beta(f,m(f))) \geq_f (w, \beta(f,w))$.
Clearly $(w,\beta(f,w)) \geq_f (w,t)$.
Hence $(m(f), \beta(f,m(f))) \geq_f (w, t)$, contrary to our assumption.
Thus (m,n) is stable. Q.E.D.

Note: The above proof is not valid if instead of assuming that preferences are lexicographic for workers, we merely assume that they are separable for them. The conflict arises since TC defines a best technique according to the preferences of the firms and not that of the workers.
It is also worth noting that Theorem 1 and its proof would continue to remain valid if the definition of a three-sided system being "lexicographic for workers" had been weaker than what we insist in this paper. Thus if we do not insist on a three-sided system to be separable in order to be lexicographic for workers, the above theorem would continue to be valid with a minor re-wording of the above proof. The alternative definition of a three-sided system being lexicographic for workers could read as: for all $w \in W$ there exists linear orders P_w on F such that for all $f,f' \in F$ with $f \neq f'$ and $t,t' \in T$: fP_wf' implies $(f,t) >_w (f',t')$.

The following example shows that if a three-sided system is merely lexicographic for workers then the existence of a stable matching is not guaranteed.

Example 1: Let $W = \{w_1,w_2\}$, $F = \{f_1,f_2\}$, $T = \{t_1, t_2\}$.
Assume that the system is lexicographic for workers with both w_1 and w_2 preferring t_1 to t_2 for any given firm f. Suppose that both w_1 and w_2 prefer f_1 to f_2.
Suppose f_1 prefers (w_2,t_1) to (w_1, t_1) to (w_1,t_2) to (w_2,t_2) and f_2 prefers (w_1,t_1) to (w_2,t_1) to (w_2,t_2) to (w_1, t_2).
Suppose that t_1 prefers (f_2,w_1) to (f_1,w_2) to (f_2,w_2) to (f_1,w_1) and t_2 prefers (f_1,w_1) to (f_1,w_2).

Let us consider the following four matchings:
(1) $\{(f_1 ,w_1 ,t_1), (f_2 ,w_2 ,t_2)\}$;
(2) $\{(f_1 ,w_1 ,t_2), (f_2 ,w_2 ,t_1)\}$;
(3) $\{(f_2 ,w_1 ,t_1), (f_1 ,w_2 ,t_2)\}$;
(4) $\{(f_2 ,w_1 ,t_2), (f_1 ,w_2 ,t_1)\}$.
Matching (1) is blocked by (f_2, w_2, t_1) since w_2 prefers (f_2,t_1) to (f_2,t_2), f_2 prefers (w_2, t_1) to (w_2, t_2) and t_1 prefers (f_2,w_2) to (f_1,w_1).
Matching (2) is blocked by (f_1, w_2,t_1) since w_2 prefers (f_1,t_1) to (f_2,t_1), f_1 prefers (w_2,t_1) to (w_1,t_2) and t_1 prefers (f_1,w_2) to (f_2,w_2).
Matching (3) is blocked by (f_1,w_1,t_2) since w_1 prefers (f_1,t_2) to (f_2,t_2), f_1 prefers (w_1,t_2) to (w_2,t_2) and t_2 prefers (f_1,w_1) to (f_1,w_2).
Matching (4) is blocked by (f_2,w_1,t_1) since w_1 prefers (f_2,t_1) to (f_2,t_2), f_2 prefers (w_1,t_1) to (w_1,t_2) and t_1 prefers (f_2,w_1) to (f_1,w_2).
Hence none of the four matchings are stable.
Further, $\beta(f_1,w_2) = \beta(f_2,w_1) = t_1$. This contradicts TS.

It is worth noting that TS is not necessary for the existence of a stable matching for a three-sided system, as the following example reveals.

Example 2: Let $W = \{w_1,w_2,w_3\}$, $F= \{f_1,f_2,f_3\}$ and $T = \{t_1,t_2,t_3\}$.
Suppose that for each $w\in W$ there exists a linear order P_w on F satisfying $f_1P_wf_2P_wf_3$ and for each $f\in F$ there exists a linear order P_f on W satisfying $w_1P_fw_2P_fw_3$. Suppose for each $w\in W$ there exists a

linear order Q_w on T and for each f∈F there exists a linear order Q_f on T. Suppose $t_1 Q_a t_2 Q_a t_3$ for $a \in \{f_1, w_1, w_2\}$ and $t_3 Q_a t_2 Q_a t_1$ for $a \in \{w_3, f_2, f_3\}$. Further suppose that for all w,w'∈W, f,f' ∈F and t,t'∈T with w≠w', f≠f' and t≠t': (a) $(w,t) >_f (w',t')$ if and only if $wP_f w_1$; (b) $(w,t) >_f (w,t')$ if and only if $tQ_f t'$; (c) $(f,t) >_w (f',t')$ if and only if $fP_w f'$; (d) $(f,t) >_w (f,t')$ if and only if $tQ_w t'$.
In addition suppose that for all t∈T, f'∈F, w'∈W and i∈{1,2,3}: $(f_i,w_i) \geq_t (f',w')$ if and only if $t = t_i$.
Towards a contradiction suppose that this system satisfies TS.
Then there exists a function β:F× W→T such that (a) for all w,w'∈W and f,f' ∈F with w≠w' and f≠f':$\beta(f,w) \neq \beta(f',w')$; (b) for all w∈W and f∈F: $[(w,\beta(f,w)) \geq_f (w,t)$ for all t∈T]. Thus, $\beta(f_1,w_1) = t_1$ and $\beta(f_3,w_3) = t_3$. Since $\beta(f_2,w_2) \in \{t_1,t_3\}$, the requirements of TS are violated. Thus this system does not satisfy TS.
However, the matching with the associated triplets being (w_i,f_i,t_i) for i = 1,2,3 is indeed a stable matching.

A three-sided system $[\{\geq_f:f\in F\}, \{\geq_w:w\in W\}, \{\geq_t:t\in T\}]$ is said to satisfy **Agreement over Workers** if for f∈F, t∈T and w'∈W: $(w,t) >_f (w', t)$ implies $(f,w) >_t (f, w')$.

Theorem 2: Suppose a three-sided system is strongly separable (i.e. separable for firms and lexicographic for workers) and satisfies Agreement over Workers. Then there exists a stable matching.

Proof: Suppose that for all w∈W, there exists linear orders P_w on F and Q_w on T such that: (a) for all f,f'∈F with f ≠ f' and t,t'∈T: $fP_w f'$ implies $(f,t) >_w (f',t')$; (b) for all f∈F and t,t'∈T with t ≠ t': $tQ_w t'$ implies $(f,t) >_w (f,t')$.
Suppose in addition that for all f∈F, there exists a linear order P_f on W and Q_f on T such that for all w,w'∈F with w ≠ w' and t,t'∈T with t ≠ t': $wP_f w'$ implies $(w,t) >_f (w',t)$ and $tP_f t'$ implies $(w,t) >_f (w,t')$.

Consider the two-sided matching model based on F and W where for each $f \in F$ and $w \in W$, preferences are given by P_f and P_w respectively. As in Gale and Shapley (1962), we get a job-matching m that is stable, i.e. for all $w \in W$ and $f \in F$: either $m(f)P_f w$ or $m^{-1}(w)P_w f$.
For $t \in T$, let P_t be a linear order on F such that for all $f,f' \in F$ with $f \neq f'$: $fP_t f'$ if and only if $(f, m(f)) >_t (f', m(f'))$.
Consider the two-sided matching model based on F and T where for each $f \in F$ and $t \in T$, preferences are given by Q_f and P_t respectively. As in Gale and Shapley (1962), we get a technique-matching n such that for all $t \in T$ and $f \in F$: either $n(f)Q_f t$ or $n^{-1}(t)P_t f$.
Towards a contradiction suppose that the matching (m,n) is not stable.
Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that: $(f,t) >_w (m^{-1}(w), n(m^{-1}(w)))$, $(w,t) >_f (m(f), n(f))$ and $(f,w) >_t (n^{-1}(t), m(n^{-1}(t)))$.
Since the preferences of workers are lexicographic with firms receiving priority over techniques, it must be either (a) $f = m^{-1}(w)$ and $tQ_w\, n(m^{-1}(w))$ or (b) $fP_w m^{-1}(w)$.
Suppose $f = m^{-1}(w)$.
Thus, $tQ_w\, n(m^{-1}(w))$.
Since preferences of firms are separable we must have $tQ_f n(f)$.
$tQ_f n(f)$ and the stability of the matching n implies $(n^{-1}(t), m(n^{-1}(t))) >_t (f,m(f))$.
Thus, $(f, w) >_t (f, m(f))$.
This contradicts $w = m(f)$.
Hence suppose $fP_w m^{-1}(w)$. By the stability of the matching m, we must have $m(f)P_f w$.
Since preferences of firms are separable given $(w,t) >_f (m(f), n(f))$, the fact that we have $m(f)P_f w$ implies $tQ_f n(f)$.
$tQ_f n(f)$ and the stability of the matching n implies $(n^{-1}(t), m(n^{-1}(t))) >_t (f,m(f))$.
Thus, $(f, w) >_t (f, m(f))$.

Since the three-sided system is assumed to satisfy agreement over workers and the preferences of firms are separable, $(f, w) >_t (f, m(f))$ implies $wP_f m(f)$, contradicting $m(f)P_f w$ as obtained earlier. Thus (m,n) is stable. Q.E.D.

In example 1 the preferences of the workers are lexicographic (with firms getting priority over technologists), but the preferences of the firms are not separable. Thus although the three-sided system satisfies agreement over workers, it does not admit a stable matching.

In the following example preferences are strongly separable but the system does not satisfy agreement over workers and does not admit a stable matching.

<u>Example 3</u>: Let $W = \{w_1,w_2\}$, $F = \{f_1,f_2\}$, $T = \{t_1, t_2\}$.
Assume that the system is lexicographic for workers (with firms receiving priority over technologists). Suppose that for any given firm both workers prefer t_2 to t_1 and that both workers prefer f_1 to f_2.
Suppose the preferences of the firms are also lexicographic (although not in the sense that we have defined in this paper), with technologists receiving priority over workers. Hence the preferences of the firms are separable. Suppose both firms prefer t_2 to t_1 and for any given technique prefer w_2 to w_1.
Suppose t_1 prefers (f_1, w_2) to (f_1, w_1).
Suppose t_2 prefers (f_2,w_2) to (f_1, w_1) and (f_1,w_1) to (f_2, w_1) to (f_1,w_2).

Let us consider the following four matchings:

(1) $\{(f_1 ,w_1 ,t_1), (f_2 ,w_2 ,t_2)\}$;
(2) $\{(f_1 ,w_1 ,t_2), (f_2 ,w_2 ,t_1)\}$;
(3) $\{(f_2 ,w_1 ,t_1), (f_1 ,w_2 ,t_2)\}$;
(4) $\{(f_2 ,w_1 ,t_2), (f_1 ,w_2 ,t_1)\}$.

Matching (1) is blocked by (f_1, w_2, t_1) since w_2 prefers (f_1,t_1) to (f_2,t_2), f_1 prefers (w_2, t_1) to (w_1, t_1) and t_1 prefers (f_1,w_2) to (f_1,w_1).
Matching (2) is blocked by (f_2, w_2,t_2) since w_2 prefers (f_2,t_2) to (f_2,t_1), f_2 prefers (w_2,t_2) to (w_2,t_1) and t_2 prefers (f_2,w_2) to (f_1,w_1).
Matching (3) is blocked by (f_2,w_1,t_2) since w_1 prefers (f_2,t_2) to (f_2,t_1), f_2 prefers (w_1,t_2) to (w_2,t_1) and t_2 prefers (f_2,w_1) to (f_1,w_2).
Matching (4) is blocked by (f_1,w_1,t_2) since w_1 prefers (f_1,t_2) to (f_2,t_2), f_1 prefers (w_1,t_2) to (w_2,t_1) and t_2 prefers (f_1,w_1) to (f_2,w_1).
Hence none of the four matchings are stable.
Note that given t_2, f_1 prefers w_2 to w_1, whereas given f_1, t_2 prefers w_1 to w_2.
Hence the system does not satisfy agreement over workers.

It is instructive to note that in example 3, although the preferences of the workers and firms are both lexicographic (although not in the sense in which it is defined here) with workers giving priority to firms over techniques, a stable matching does not exist, since firms accord priority to techniques over workers. The reciprocation of priority between firms and workers that was assumed by Danilov (2003) is absent in example 3.

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